A NEW GENERALIZATION OF THE TRAPEZOID FORMULA FOR n-TIME DIFFERENTIABLE MAPPINGS AND APPLICATIONS

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Abstract. A new generalization of the trapezoid formula for n-time differentiable mappings and applications in Numerical Analysis are given.

1. Introduction

In the recent paper [1], P. Cerone, S.S. Dragomir and J. Roumeliotis proved the following generalization of the trapezoid rule.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have the equality*

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] + \int_a^b T_n(t) f^{(n)}(t)\,dt, \tag{1.1}$$

where

$$T_n(t) := \frac{1}{n!} \left[\frac{(b-t)^n + (-1)^n (t-a)^n}{2} \right], \qquad t \in [a,b]. \tag{1.2}$$

In the same paper, the authors pointed out the following inequality which provides an approximation formula for the integral $\int_a^b f(t)\,dt$ whose error can be estimated in terms of the sup-norm of $f^{(n)}(t)$.

Corollary 1. *Under the above assumptions, we have the inequality*

$$\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \le \frac{(b-a)^{n+1}}{(n+1)!} \left\| f^{(n)} \right\|_\infty \times \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^{2r+1}-1}{2^r} & \text{if } n = 2r+1 \end{cases}. \tag{1.3}$$

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If, in the above corollary, we consider $n = 1$, then we get the known inequality [2]

$$\left| \int_a^b f(t)\,dt - \frac{f(a) + f(b)}{2}(b-a) \right| \leq \frac{1}{2}(b-a)^2 \|f'\|_\infty . \tag{1.4}$$

For $n = 2$, we obtain

$$\left| \int_a^b f(t)\,dt - \frac{f(a) + f(b)}{2}(b-a) - \frac{(b-a)^2}{2} \cdot \frac{f'(a) + f'(b)}{2} \right| \tag{1.5}$$
$$\leq \frac{(b-a)^3}{6} \left\| f'' \right\|_\infty .$$

For other recent results concerning the trapezoid formula, see the book [11] and the recent papers [1]-[10] and [12]-[13], where further references are given.

The main aim of this paper is to point out a generalization of the trapezoid rule and inequality in a different way. Applications in Numerical Analysis for quadrature formulae will also be provided. A perturbed trapezoidal type rule is presented in Section 4 in which a number of *premature* results are given that provide tighter bounds than the traditional Grüss, Chebychev and Lupas inequalities.

2. Integral Identities

We start with the following result.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ $(n \geq 1)$ is absolutely continuous on $[a,b]$. Then*

$$\int_a^b f(t)\,dt \tag{2.1}$$
$$= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right]$$
$$+ \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t)\,dt,$$

for all $x \in [a,b]$.

Proof. The proosf is by mathematical induction.

For $n = 1$, we have to prove that

$$\int_a^b f(t)\,dt = (x-a) f(a) + (b-x) f(b) + \int_a^b (x-t) f^{(1)}(t)\,dt, \tag{2.2}$$

which is straightforward as may be seen by the integration by parts formula applied for the integral

$$\int_a^b (x-t) f^{(1)}(t)\,dt.$$

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we wish to show that:

$$\int_a^b f(t)\,dt \tag{2.3}$$
$$= \sum_{k=0}^{n} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right]$$
$$+\frac{1}{(n+1)!}\int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt.$$

For this purpose, we apply formula (2.2) for the mapping $g(t) := (x-t)^n f^{(n)}(t)$, which is absolutely continuous on $[a,b]$, and then, we can write:

$$\int_a^b (x-t)^n f^{(n)}(t)\,dt \tag{2.4}$$
$$= (x-a)(x-a)^n f^{(n)}(a) + (b-x)(x-b)^n f^{(n)}(b)$$
$$+\int_a^b (x-t)\frac{d}{dt}\left[(x-t)^n f^{(n)}(t)\right]dt$$
$$= \int_a^b (x-t)\left[-n(x-t)^{n-1} f^{(n)}(t) + (x-t)^n f^{(n+1)}(t)\right]dt$$
$$+(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b)$$
$$= -n\int_a^b (x-t)^n f^{(n)}(t)\,dt + \int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt$$
$$+(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n+1)}(t)\,dt.$$

From this identity we can get

$$\int_a^b (x-t)^n f^{(n)}(t)\,dt$$
$$= \frac{1}{n+1}\int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt$$
$$+\frac{1}{n+1}\left[(x-a)^{n+1} f^{(n)}(a) + (-1)^n (b-x)^{n+1} f^{(n)}(b)\right].$$

Now, using the induction hypothesis, we have

$$\int_a^b f(t)\,dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] \quad +$$

$$
\begin{aligned}
&+\frac{1}{n!}\left[\frac{1}{n+1}\int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt\right. \\
&\left.+\frac{1}{n+1}\left[(x-a)^{n+1} f^{(n)}(a)+(b-x)^{n+1} f^{(n)}(b)\right]\right] \\
= &\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a)+(-1)^k (b-x)^{k+1} f^{(k)}(b)\right] \\
&+\frac{1}{(n+1)!}\int_a^b (x-t)^{n+1} f^{(n+1)}(t)\,dt
\end{aligned}
$$

and the identity (2.3) is proved. This completes the proof. ∎

The following corollary is useful in practice.

Corollary 2. *With the above assumptions for f and R, we have the particular identities (which can also be obtained by using Taylor's formula with the integral remainder)*

$$
\begin{aligned}
\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}(b-a)^{k+1} f^{(k)}(b) \\
&+\frac{(-1)^n}{n!}\int_a^b (t-a)^n f^{(n)}(t)\,dt,
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}(b-a)^{k+1} f^{(k)}(a) \\
&+\frac{1}{n!}\int_a^b (b-t)^n f^{(n)}(t)\,dt,
\end{aligned} \tag{2.6}
$$

and the identity (see also [13]*)*

$$
\begin{aligned}
\int_a^b f(t)\,dt &= \sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right] \\
&+\frac{(-1)^n}{n!}\int_a^b \left(t-\frac{a+b}{2}\right)^n f^{(n)}(t)\,dt.
\end{aligned} \tag{2.7}
$$

Remark 1. a) *For $n=1$, we get the identity (2.2) which is a generalization of the trapezoid rule.*

i) *For $x=a$ in (2.2), we capture the "right rectangle rule"*

$$\int_a^b f(t)\,dt=(b-a)\,f(b)-\int_a^b (t-a)\,f'(t)\,dt.$$

ii) *For $x=b$ in (2.2), we obtain the "left rectangle rule"*

$$\int_a^b f(t)\,dt = (b-a) f(a) - \int_a^b (b-t) f'(t)\,dt. \tag{2.8}$$

iii) *Finally, for $x=\frac{a+b}{2}$, we get* [2]

$$\int_a^b f(t)\,dt = \frac{f(a)+f(b)}{2}(b-a) - \int_a^b \left(t-\frac{a+b}{2}\right) f'(t)\,dt \tag{2.9}$$

which is the "trapezoid rule".

b) *For $n=2$, we get the identity:*

$$\begin{aligned} &\int_a^b f(t)\,dt \\ = \; & (x-a) f(a) + (b-x) f(b) \\ & + \frac{1}{2}\left[(x-a)^2 f'(a) + (b-x)^2 f'(b)\right] + \frac{1}{2}\int_a^b (x-t)^2 f''(t)\,dt. \end{aligned} \tag{2.10}$$

i) *If in (2.10) we choose $x=b$, then we obtain the "perturbed left rectangle rule"*

$$\int_a^b f(t)\,dt = (b-a) f(a) + \frac{1}{2}(b-a)^2 f'(a) + \frac{1}{2}\int_a^b (t-a)^2 f''(t)\,dt, \tag{2.11}$$

which can also be obtained by using Taylor's formula with the integral remainder.

ii) *If in (2.10) we choose $x=a$, we can write the "perturbed right rectangle rule"*

$$\int_a^b f(t)\,dt = (b-a) f(b) - \frac{1}{2}(b-a)^2 f'(b) + \frac{1}{2}\int_a^b (t-b)^2 f''(t)\,dt. \tag{2.12}$$

iii) *Finally, for $x=\frac{a+b}{2}$, we capture the "perturbed trapezoid rule"* [13]

$$\begin{aligned} \int_a^b f(t)\,dt \; = \; & \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{8}\left(f'(a) - f'(b)\right) \\ & + \frac{1}{2}\int_a^b \left(t-\frac{a+b}{2}\right)^2 f''(t)\,dt. \end{aligned} \tag{2.13}$$

3. Integral Inequalities

Using the integral representation by Theorem 1, we can prove the following inequality

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a mapping so that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on $[a,b]$. Then*

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right| \tag{3.1}$$
$$\leq\begin{cases}\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right] & \text{if} \quad f^{(n)}\in L_\infty[a,b]\,;\\ \frac{\left\|f^{(n)}\right\|_p}{n!}\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1\\ & \text{and} \quad f^{(n)}\in L_p[a,b]\,;\\ \frac{\left\|f^{(n)}\right\|_1}{n!}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^n & \end{cases}$$

for all $x\in[a,b]$.

Proof. Using the representation (2.1) and the properties of the modulus, we have

$$\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right]\right|$$
$$\leq \frac{1}{n!}\int_a^b |x-t|^n\left|f^{(n)}(t)\right|dt=:R.$$

Observe that

$$\begin{aligned}R &\leq \left[\frac{1}{n!}\int_a^b|x-t|^n\,dt\right]\left\|f^{(n)}\right\|_\infty\\ &= \frac{\left\|f^{(n)}\right\|_\infty}{n!}\left[\int_a^x (x-t)^n\,dt+\int_x^b (t-x)^n\,dt\right]\\ &= \frac{\left\|f^{(n)}\right\|_\infty}{n!}\left[\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n+1}\right]\end{aligned}$$

and the first inequality in (3.1) is proved.

Using Hölder's integral inequality, we also have

$$\begin{aligned}R &\leq \frac{1}{n!}\left(\int_a^b\left|f^{(n)}(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^b|x-t|^{nq}\,dt\right)^{\frac{1}{q}}\\ &= \frac{1}{n!}\left\|f^{(n)}\right\|_p\left[\frac{(x-a)^{nq+1}+(b-x)^{nq+1}}{nq+1}\right]^{\frac{1}{q}},\end{aligned}$$

which proves the second inequality in (3.1).

Finally, let us observe that

$$
\begin{aligned}
R &\leq \frac{1}{n!}\sup_{t\in[a,b]}|x-t|^n\int_a^b\left|f^{(n)}(t)\right|dt \\
&= \frac{1}{n!}\left[\sup_{t\in[a,b]}|x-t|\right]^n\left\|f^{(n)}\right\|_1 \\
&= \frac{1}{n!}\left[\max(x-a,b-x)\right]^n\left\|f^{(n)}\right\|_1 \\
&= \frac{1}{n!}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^n\left\|f^{(n)}\right\|_1
\end{aligned}
$$

and the theorem is completely proved. ∎

The following corollary is useful in practice.

Corollary 3. *With the above assumptions for f and n, we have the particular inequalities*

$$
\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{(-1)^k}{(k+1)!}(b-a)^{k+1}f^{(k)}(b)\right| \tag{3.2}
$$

$$
\leq M:=\begin{cases}
\frac{\left\|f^{(n)}\right\|_\infty}{(n+1)!}(b-a)^{n+1} & \text{if} \quad f^{(n)}\in L_\infty[a,b]; \\[2mm]
\frac{\left\|f^{(n)}\right\|_p}{n!}\frac{(b-a)^{n+\frac{1}{q}}}{(nq+1)^{1/q}} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\
 & \text{and} \quad f^{(n)}\in L_p[a,b]; \\[2mm]
\frac{\left\|f^{(n)}\right\|_1}{n!}(b-a)^n, &
\end{cases}
$$

and

$$
\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}(b-a)^{k+1}f^{(k)}(b)\right|\leq M
$$

and (see also [13]*)*

$$
\left|\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right| \tag{3.3}
$$

$$
\leq\begin{cases}
\frac{\left\|f^{(n)}\right\|_\infty}{2^n(n+1)!}(b-a)^{n+1} & \text{if} \quad f^{(n)}\in L_\infty[a,b]; \\[2mm]
\frac{\left\|f^{(n)}\right\|_p}{2^n n!(nq+1)^{1/q}}(b-a)^{n+\frac{1}{q}} & \text{if} \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\
 & \text{and} \quad f^{(n)}\in L_p[a,b]; \\[2mm]
\frac{\left\|f^{(n)}\right\|_1}{2^n n!}(b-a)^n; &
\end{cases}
$$

respectively.

Remark 2. *If we put $n = 1$ in (3.1), we capture the inequality*

$$\left| \int_a^b f(t)\,dt - (x-a) f(a) + (b-x) f(b) \right| \tag{3.4}$$
$$\leq \begin{cases} \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 \right] \left\| f^{(1)} \right\|_\infty & \text{if} \quad f' \in L_\infty[a,b]; \\ \left\| f' \right\|_p \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} & \text{if} \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ & \text{and} \quad f' \in L_p[a,b]; \\ \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \left\| f' \right\|_1; \end{cases}$$

for all $x \in [a,b]$, and, in particular,

a) *the "left rectangle" inequality*

$$\left| \int_a^b f(t)\,dt - (b-a) f(a) \right| \leq \begin{cases} \frac{\|f'\|_\infty}{2}(b-a)^2 & \text{if } f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a,b]; \\ \|f'\|_1 (b-a). \end{cases}$$

b) *the "right rectangle" inequality*

$$\left| \int_a^b f(t)\,dt - (b-a) f(b) \right| \leq \begin{cases} \frac{\|f'\|_\infty}{2}(b-a)^2 & \text{if } f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a,b]; \\ \|f'\|_1 (b-a). \end{cases}$$

c) *the "trapezoid" inequality*

$$\left| \int_a^b f(t)\,dt - \frac{f(a) + f(b)}{2}(b-a) \right| \tag{3.5}$$
$$\leq \begin{cases} \frac{\|f'\|_\infty}{4}(b-a)^2 & \text{if } f' \in L_\infty[a,b]; \\ \frac{\|f'\|_p}{2(q+1)^{1/q}}(b-a)^{1+\frac{1}{q}} & \text{if } f' \in L_p[a,b]; \\ \frac{\|f'\|_1}{2}(b-a). \end{cases}$$

Remark 3. *If we put $n = 2$ in (3.1), we get the inequality*

$$\left|\int_a^b f(t)\,dt - (x-a) f(a) - (b-x) f(b) - \frac{1}{2}\left[(x-a)^2 f'(a) - (b-x)^2 f'(b)\right]\right| \tag{3.6}$$

$$\leq \begin{cases} \frac{\left\|f''\right\|_\infty}{6}\left[(b-a)^3 + (b-x)^3\right] & \text{if} \quad f'' \in L_\infty[a,b]; \\ \frac{\left\|f''\right\|_p}{2}\left[\frac{(x-a)^{2q+1}+(b-x)^{2q+1}}{2q+1}\right]^{\frac{1}{q}} & \text{if} \quad p>1, \frac{1}{p}+\frac{1}{q}=1 \\ & \text{and} \quad f'' \in L_p[a,b]; \\ \frac{\left\|f''\right\|_1}{2}\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2; \end{cases}$$

for all $x \in [a,b]$, and, in particular:

a) *the "perturbed left rectangle" inequality*

$$\left|\int_a^b f(t)\,dt - (b-a) f(a) - \frac{1}{2}(b-a)^2 f'(a)\right| \tag{3.7}$$

$$\leq \quad M_2 := \begin{cases} \frac{\|f''\|_\infty}{6}(b-a)^3 & \text{if } f'' \in L_\infty[a,b]; \\ \frac{\left\|f''\right\|_p}{2(2q+1)^{1/q}}(b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a,b]; \\ \frac{\left\|f''\right\|_1}{2}(b-a)^2; \end{cases}$$

b) *the "perturbed right rectangle" inequality*

$$\left|\int_a^b f(t)\,dt - (b-a) f(b) + \frac{1}{2}(b-a)^2 f'(b)\right| \leq M_2 \tag{3.8}$$

c) *the "perturbed trapezoid" inequality*

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{(b-a)^2}{8}\left(f'(b) - f'(a)\right)\right| \tag{3.9}$$

$$\leq \begin{cases} \frac{\left\|f''\right\|_\infty}{24}(b-a)^3 & \text{if } f'' \in L_\infty[a,b]; \\ \frac{\left\|f''\right\|_p}{8(2q+1)^{1/q}}(b-a)^{2+\frac{1}{q}} & \text{if } f'' \in L_p[a,b]; \\ \frac{\left\|f''\right\|_1}{8}(b-a)^2. \end{cases}$$

4. A Perturbed Version

A *premature* Grüss inequality is embodied in the following lemma which was proven by Cerone and Dragomir in the papers [14] and [16].

Lemma 1. *Let f, g be integrable functions defined on $[a,b]$ and let $d \leq g(t) \leq D$. Then*

$$|T(f,g)| \leq \frac{D-d}{2}[T(f,f)]^{\frac{1}{2}}, \tag{4.1}$$

where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt.$$

Using the above lemma, the following result may be stated.

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ so that the derivative $f^{(n-1)}$, $n \geq 1$ is absolutely continuous on $[a,b]$. Assume that there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \leq f^{(n)}(t) \leq \Gamma$ a.e on $[a,b]$. Then, the following inequality holds*

$$\begin{aligned}(4.2)\qquad |P_T(x)| &:= \Bigg| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1}\bigg(\frac{1}{(k+1)!}\times \\ &\Big[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\Big]\bigg) \\ &- \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!}\left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right]\Bigg| \\ &\leq \frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!} I(x,n) \\ &\leq \frac{\Gamma-\gamma}{2}\cdot\frac{n}{n+1}\cdot\frac{(b-a)^{n+1}}{\sqrt{2n+1}},\end{aligned}$$

where

$$\begin{aligned}(4.3)\quad I(x,n) &= \frac{1}{(n+1)\sqrt{2n+1}}\Big\{n^2(b-a)\Big[(x-a)^{2n+1} + (b-x)^{2n+1}\Big] \\ &\quad + (2n+1)(x-a)(b-x)\left[(x-a)^n - (x-b)^n\right]^2\Big\}^{\frac{1}{2}}.\end{aligned}$$

Proof. Applying the premature Grüss result (4.1) on $(x-t)^n$ and $f^{(n)}(t)$, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^n\,dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right| \\ \leq\ & \frac{\Gamma-\gamma}{2}\left\{\frac{1}{b-a}\int_a^b (x-t)^{2n}\,dt - \left[\frac{1}{b-a}\int_a^b (x-t)^n\,dt\right]^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b (x-t)^n f^{(n)}(t)\, dt \right. \\ & \left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)(b-a)} \cdot \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right| \\ \leq\ & \frac{\Gamma-\gamma}{2} \left\{ \frac{(x-a)^{2n+1} + (b-x)^{2n+1}}{(2n+1)(b-a)} - \left[\frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(b-a)(n+1)} \right]^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

Further simplification of the above result by multiplying throughout by $\frac{b-a}{n!}$ gives

$$\begin{aligned} & \left| \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t)\, dt \right. \\ & \left. - \frac{(x-a)^{n+1} + (-1)^n (b-x)^{n+1}}{(n+1)!} \cdot \left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} \right] \right| \\ \leq\ & \frac{\Gamma-\gamma}{2} \cdot \frac{1}{n!} J(x,n), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} J^2(x,n) = & \frac{1}{(2n+1)(n+1)^2} \left\{ (n+1)^2 (A+B) \left(A^{2n+1} + B^{2n+1} \right) \right. \\ & \left. - (2n+1) \left(A^{n+1} + (-1)^n B^{n+1} \right)^2 \right\} \end{aligned} \tag{4.5}$$

with $A = x-a$, $B = b-x$.

Now, from (4.5),

$$\begin{aligned} & (2n+1)(n+1)^2 J^2(x,n) \\ = & n^2 (A+B) \left(A^{2n+1} + B^{2n+1} \right) \\ & + (2n+1) \left[(A+B) \left(A^{2n+1} + B^{2n+1} \right) - \left(A^{n+1} + (-1)^n B^{n+1} \right)^2 \right] \\ = & n^2 (A+B) \left(A^{2n+1} + B^{2n+1} \right) \\ & + (2n+1) \left[AB \left(A^{2n} + B^{2n} \right) - 2A^{n+1} \cdot (-1)^n B^{n+1} \right] \\ = & n^2 (A+B) \left[A^{2n+1} + B^{2n+1} \right] + (2n+1) AB \left[A^n - (-B)^n \right]^2 \end{aligned}$$

Now, substitution of $A = x-a$, $B = b-x$ and the fact that $A+B = b-a$ gives $I(x,n) = \frac{J(x,n)}{(n+1)\sqrt{2n+1}}$, as presented in (4.3). Substitution of identity (2.1) into (4.4) gives (4.2) and the first part of the theorem is thus proved.

The upper bound is obtained by taking either $I(a,n)$ or $I(b,n)$ since $I(x,n)$ is convex. Hence the theorem is completely proved. ∎

Corollary 4. *Let the conditions of Theorem 4 hold. Then the following result holds*

$$\begin{aligned}(4.6)\quad &\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}\left[f^{(k)}(a)+(-1)^k f^{(k)}(b)\right]\right.\\ &\left.-\left(\frac{b-a}{2}\right)^n\frac{[1+(-1)^n]}{(n+1)!}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]\right|\\ \leq\ &\frac{\Gamma-\gamma}{2}\cdot\frac{1}{n!}\left(\frac{b-a}{2}\right)^{n+1}\cdot\frac{1}{\sqrt{2n+1}}\cdot\begin{cases}\frac{2n}{n+1}, & n\text{ even}\\ 2, & n\text{ odd}\end{cases}.\end{aligned}$$

Proof. Taking $x=\frac{a+b}{2}$ in (4.2) gives (4.2), where

$$I\left(\frac{a+b}{2},n\right)=\frac{1}{(n+1)\sqrt{2n+1}}\left(\frac{b-a}{2}\right)^{n+1}\left\{4n^2+(2n+1)\left[1+(-1)^n\right]^2\right\}^{\frac{1}{2}}.$$

Examining the above expression for n even or n odd readily gives the result (4.6). ∎

Remark 4. *For n even, then the third term in the modulus sign vanishes and thus there is no perturbation to the trapezoidal rule (4.6).*

Theorem 5. *Let the conditions of Theorem 4 be satisfied. Further, suppose that $f^{(n)}$ is differentiable and be such that*

$$\left\|f^{(n+1)}\right\|_\infty:=\sup_{t\in[a,b]}\left|f^{n+1}(t)\right|<\infty.$$

Then

$$|P_T(x)|\leq\frac{b-a}{\sqrt{12}}\left\|f^{(n+1)}\right\|_\infty\cdot\frac{1}{n!}I(x,n),\tag{4.7}$$

where $P_T(x)$ is the perturbed trapezoidal type rule given by the left hand side of (4.2) and $I(x,n)$ is as given by (4.3).

Proof. Let $f,g:[a,b]\to\mathbb{R}$ be absolutely continuous and f',g' be bounded. Then Chebychev's inequality holds (see [15, p. 207])

$$|T(f,g)|\leq\frac{(b-a)^2}{12}\sup_{t\in[a,b]}|f'(t)|\cdot\sup_{t\in[a,b]}|g'(t)|.$$

Matić, Pečarić and Ujević [16] using a *premature* Grüss type argument proved that

$$|T(f,g)|\leq\frac{(b-a)}{\sqrt{12}}\sup_{t\in[a,b]}|g'(t)|\sqrt{T(f,f)}.\tag{4.8}$$

Thus, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $(x-t)^n$ with f in (4.8) readily produces (4.7) where $I(x,n)$ is as given by (4.3). ∎

Theorem 6. *Let the conditions of Theorem 4 be satisfied. Further, suppose that $f^{(n)}$ is locally absolutely continuous on (a,b) and let $f^{(n+1)}\in L_2(a,b)$. Then*

$$|P_T(x)|\leq\frac{b-a}{\pi}\left\|f^{(n+1)}\right\|_2\cdot\frac{1}{n!}I(x,n),\tag{4.9}$$

where $P_T(x)$ is the perturbed trapezoidal type rule given by the left hand side of (4.2) and $I(x,n)$ is as given in (4.3).

Proof. The following result was obtained by Lupaş (see [15, p. 210]). For $f, g : (a, b) \to \mathbb{R}$ locally absolutely continuous on (a, b) and $f', g' \in L_2(a, b)$, then

$$|T(f,g)| \leq \frac{(b-a)^2}{\pi^2} \|f'\|_2 \|g'\|_2,$$

where

$$\|h\|_2 := \left(\frac{1}{b-a} \int_a^b |h(t)|^2 \right)^{\frac{1}{2}} \quad \text{for} \quad h \in L_2(a,b).$$

Matić, Pečarić and Ujević [16] further show that

$$|T(f,g)| \leq \frac{(b-a)}{\pi} \|g'\|_2 \sqrt{T(f,f)}. \tag{4.10}$$

Now, associating $f^{(n)}(\cdot)$ with $g(\cdot)$ and $(x-t)^n$ with f in (4.10) gives (4.9), where $I(x,n)$ is found in (4.3). ∎

Remark 5. *Results (4.7) and (4.9) are not readily comparable to that obtained in Theorem 4 since the bound now involves the behaviour of $f^{(n+1)}(\cdot)$ rather than $f^{(n)}(\cdot)$.*

5. Application In Numerical Integration

Consider the partition $I_m : a = x_0 < x_1 < ... < x_{m-1} < x_m = b$ of the interval $[a, b]$ and the intermediate points $\xi = (\xi_0, ..., \xi_{m-1})$, where $\xi_j \in [x_j, x_{j+1}]$ $(j = 0, ..., m-1)$. Put $h_j := x_{j+1} - x_j$ and $\vartheta(h) = \max\{h_j | j = 0, ..., m-1\}$.

In [1], the authors considered the following generalization of the trapezoid formula

$$T_{m,n}(f, I_m) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{h_j^{k+1}}{(k+1)!} \left[\frac{f^{(k)}(x_j) + (-1)^k f^{(k)}(x_{j+1})}{2} \right] \tag{5.1}$$

and proved the following theorem:

Theorem 7. *Let $f : [a, b] \to \mathbb{R}$ be such that it's derivative $f^{(n-1)}$ is absolutely continuous on $[a, b]$. Then we have*

$$\int_a^b f(t)\,dt = T_{m,n}(f, I_m) + R_{m,n}(f, I_m), \tag{5.2}$$

where the reminder $R_{m,n}(f, I_m)$ satisfies the estimate

$$|R_{m,n}(f, I_m)| \leq \frac{C_n}{(n+1)!} \left\| f^{(n)} \right\|_\infty \sum_{j=0}^{m-1} h_j^{n+1}, \tag{5.3}$$

and

$$C_n := \begin{cases} 1 & \text{if } n = 2r \\ \frac{2^{2r+1}-1}{2^{2r}} & \text{if } n = 2r+1. \end{cases}$$

Now, let us define the even more generalized quadrature formula

$$\begin{aligned}\tilde{T}_{m,n}(f,\xi,I_m) &:= \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\Big[\left(\xi_j - x_j\right)^{k+1} f^{(k)}(x_j) \\ &\quad + (-1)^k\left(x_{j+1}-\xi_j\right)^{k+1} f^{(k)}(x_{j+1})\Big],\end{aligned}$$

where x_j, ξ_j $(j = 0,...,m-1)$ are as above.
The following theorem holds.

Theorem 8. *Let f be as in Theorem 7. Then we have the formula*

$$\int_a^b f(t)\,dt = \tilde{T}_{m,n}(f,\xi,I_m) + \tilde{R}_{m,n}(f,\xi,I_m), \tag{5.4}$$

where the reminder satisfies the estimate

$$\left|\tilde{R}_{m,n}(f,\xi,I_m)\right| \tag{5.5}$$

$$:= \begin{cases} \frac{1}{(n+1)!}\left\|f^{(n)}\right\|_\infty \sum_{j=0}^{m-1}\left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1}-\xi_j\right)^{n+1}\right], \\ \\ \frac{1}{n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p \left[\sum_{j=0}^{m-1}\left(\xi_j - x_j\right)^{nq+1} + \sum_{j=0}^{m-1}\left(x_{j+1}-\xi_j\right)^{nq+1}\right]^{\frac{1}{q}}, \\ \\ \frac{1}{n!}\left\|f^{(n)}\right\|_1\left[\frac{1}{2}\vartheta(h) + \max_{j=0,\dots,m-1}\left|\xi_j - \frac{x_j+x_{j+1}}{2}\right|\right]^n. \end{cases}$$

Proof. Apply the inequality (3.1) on the subinterval $[x_j, x_{j+1}]$ to get

$$\begin{aligned}&\Bigg|\int_{x_j}^{x_{j+1}} f(t)\,dt - \sum_{k=0}^{n-1}\frac{1}{(k+1)!} \\ &\times\left[\left(\xi_j - x_j\right)^{k+1} f^{(k)}(x_j) + (-1)^k\left(x_{j+1}-\xi_j\right)^{k+1} f^{(k)}(x_{j+1})\right]\Bigg| \\ &\le \begin{cases} \frac{1}{(n+1)!}\sup_{t\in[x_j,x_{j+1}]}\left|f^{(n)}(t)\right|\left[\left(\xi_j - x_j\right)^{n+1} + \left(x_{j+1}-\xi_j\right)^{n+1}\right], \\ \\ \frac{1}{n!}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|^p ds\right)^{\frac{1}{p}}\left[\frac{\left(\xi_j-x_j\right)^{nq+1}+\left(x_{j+1}-\xi_j\right)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}, \\ \\ \frac{1}{n!}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right| ds\right)\left[\frac{1}{2}h_j + \left|\xi_j - \frac{x_j+x_{j+1}}{2}\right|\right]^n. \end{cases}\end{aligned}$$

Summing over j from 0 to $m-1$ and using the generalized triangle inequality, we have

$$\begin{aligned}
& \left|\tilde{R}_{m,n}\left(f,\xi,I_m\right)\right| \\
\leq & \left|\sum_{j=0}^{m-1}\int_{x_j}^{x_{j+1}} f\left(t\right)dt-\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\right. \\
& \left.\times\left[\left(\xi_j-x_j\right)^{k+1}f^{(k)}\left(x_j\right)+(-1)^k\left(x_{j+1}-\xi_j\right)^{k+1}f^{(k)}\left(x_{j+1}\right)\right]\right| \\
: = & \begin{cases}
\frac{1}{(n+1)!}\sum\limits_{j=0}^{m-1}\sup\limits_{t\in[x_j,x_{j+1}]}\left|f^{(n)}\left(t\right)\right|\left[\left(\xi_j-x_j\right)^{n+1}+\left(x_{j+1}-\xi_j\right)^{n+1}\right], \\
\\
\frac{1}{n!}\sum\limits_{j=0}^{m-1}\left(\int\limits_{x_j}^{x_{j+1}}\left|f^{(n)}\left(s\right)\right|^p ds\right)^{\frac{1}{p}}\left[\frac{\left(\xi_j-x_j\right)^{nq+1}+\left(x_{j+1}-\xi_j\right)^{nq+1}}{nq+1}\right]^{\frac{1}{q}}, \\
\\
\frac{1}{n!}\sum\limits_{j=0}^{m-1}\left(\int\limits_{x_j}^{x_{j+1}}\left|f^{(n)}\left(s\right)\right|ds\right)\left[\frac{1}{2}h_j+\left|\xi_j-\frac{x_j-x_{j+1}}{2}\right|\right]^n.
\end{cases}
\end{aligned}$$

As $\sup\limits_{t\in[x_j,x_{j+1}]}\left|f^{(n)}\left(t\right)\right|\leq\left\|f^{(n)}\right\|_\infty$, the first inequality is obvious.

Using the discrete Hölder inequality, we have

$$\begin{aligned}
& \frac{1}{(nq+1)^{1/q}}\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}\left(s\right)\right|^p ds\right)^{\frac{1}{p}}\left[\left(\xi_j-x_j\right)^{nq+1}+\left(x_{j+1}-\xi_j\right)^{nq+1}\right]^{\frac{1}{q}} \\
\leq & \frac{1}{(nq+1)^{1/q}}\left[\sum_{j=0}^{m-1}\left[\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}\left(s\right)\right|^p ds\right)^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}} \\
& \times\left[\sum_{j=0}^{m-1}\left[\left[\left(\xi_j-x_j\right)^{nq+1}+\left(x_{j+1}-\xi_j\right)^{nq+1}\right]^{\frac{1}{q}}\right]^q\right]^{\frac{1}{q}} \\
= & \frac{1}{(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p\left[\sum_{j=0}^{m-1}\left(\xi_j-x_j\right)^{nq+1}+\sum_{j=0}^{m-1}\left(x_{j+1}-\xi_j\right)^{nq+1}\right]^{\frac{1}{q}}
\end{aligned}$$

and the second inequality in (5.5) is proved.

Finally, let us observe that

$$\frac{1}{n!}\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}\left(s\right)\right|ds\right)\left[\frac{1}{2}h_j+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n\leq$$

$$\leq \max_{j=0,\dots,m-1}\left[\frac{1}{2}h_j+\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n\sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}}\left|f^{(n)}(s)\right|ds\right)$$
$$\leq \left[\frac{1}{2}h_j+\max_{j=0,\dots,m-1}\left|\xi_j-\frac{x_j+x_{j+1}}{2}\right|\right]^n\left\|f^{(n)}\right\|_1$$

and the last part of (5.5) is proved. ∎

Remark 6. *As $(x-a)^{\alpha}+(b-a)^{\alpha}\leq(b-a)^{\alpha}$ for $\alpha\geq 1$, $x\in[a,b]$, then we remark that the first branch of (5.5) can be bounded by*

$$\frac{1}{(n+1)!}\left\|f^{(n)}\right\|_{\infty}\sum_{j=0}^{m-1}h_j^{n+1}. \tag{5.6}$$

The second branch can be upper bounded by

$$\frac{1}{n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p\left[\sum_{j=0}^{m-1}h_j^{nq+1}\right]^{\frac{1}{q}} \tag{5.7}$$

and finally, the last branch in (5.5) can be upper bounded by

$$\frac{1}{n!}\left[\vartheta(h)\right]^n\left\|f^{(n)}\right\|_1. \tag{5.8}$$

Note that all the bounds provided by (5.6)-(5.8) are uniform bounds for $\tilde{R}_{m,n}(f,\xi,I_m)$ in terms of the intermediate points ξ.

The last inequality we can get from (5.5) is that one for which we have $\xi_j=\frac{x_j+x_{j+1}}{2}$. Consequently, we can state the following corollary (see also [13]):

Corollary 5. *Let f be as in Theorem 8. Then we have the formula*

$$\int_a^b f(t)\,dt=\tilde{T}_{m,n}(f,I_m)+\tilde{R}_{m,n}(f,I_m), \tag{5.9}$$

where

$$\tilde{T}_{m,n}(f,I_m)=\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{1}{2^{k+1}(k+1)!}\left[f^{(k)}(x_j)+(-1)^k f^{(k)}(x_{j+1})\right]h_j^{n+1} \tag{5.10}$$

and the remainder $\tilde{R}$ satisfies the estimate

$$\left|\tilde{R}_{m,n}(f,I_m)\right|\leq\begin{cases}\frac{1}{2^n(n+1)!}\left\|f^{(n)}\right\|_{\infty}\sum\limits_{j=0}^{m-1}h_j^{n+1}, \\ \\ \frac{1}{2^n n!(nq+1)^{1/q}}\left\|f^{(n)}\right\|_p\left[\sum\limits_{j=0}^{m-1}h_j^{n+1}\right]^{\frac{1}{q}}, \\ \\ \frac{1}{2^n n!}\left[\vartheta(h)\right]^n\left\|f^{(n)}\right\|_1.\end{cases}$$

Remark 7. *Similar results can be stated by using the "perturbed" versions embodied in Theorems 4, 5 and 6, but we omit the details.*

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